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(1+1) Schrödinger Lie bialgebras and their Poisson–Lie groups

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Abstract

All Lie bialgebra structures for the $(1+1)$-dimensional centrally extended Schrödinger algebra are explicitly derived and proved to be of the coboundary type. Therefore, since all of them come from a classical r-matrix, the complete family of Schrödinger Poisson–Lie groups can be deduced by means of the Sklyanin bracket. All possible embeddings of the harmonic oscillator, extended Galilei and $gl(2)$ Lie bialgebras within the Schrödinger classification are studied. As an application, new quantum (Hopf algebra) deformations of the Schrödinger algebra, including their corresponding quantum universal R-matrices, are constructed.

1 Introduction

The (1+1)-dimensional centrally extended Schrödinger algebra $\mathcal{S}$ is a non-semisimple Lie algebra spanned by the generators $\{D, H, P, K, C, M\}$ which obey the following commutation rules [1, 2]

$$\begin{array}{lll} [D, P] = -P & [D, K] = K & [K, P] = M \\ [D, H] = -2H & [D, C] = 2C & [H, C] = D \\ [K, H] = P & [K, C] = 0 & [M, \cdot\,] = 0 \\ [P, C] = -K & [P, H] = 0 & \end{array} \tag{1.1}$$

where D is the dilation, H the time translation, P the space translation, K the Galilean boost, C the conformal transformation, and M is the mass (a central generator). The Schrödinger algebra contains many remarkable Lie subalgebras: the harmonic oscillator algebra h_4 with generators $\{D, P, K, M\}$, the $gl(2)$ algebra defined by the generators $\{D, H, C, M\}$, and the $(1+1)$ extended Galilei algebra $\overline{\mathcal{G}}$ spanned by $\{H, P, K, M\}$. In turn, the three dimensional Heisenberg–Weyl algebra h_3 generated by $\{P, K, M\}$ is a subalgebra of h_4; $gl(2)$ is isomorphic to a direct sum of $sl(2, \mathbb{R})$, spanned by $\{D, H, C\}$, with the central extension M; and obviously, the $(1+1)$ Galilei algebra $\mathcal{G}$ with generators $\{H, P, K\}$ is a subalgebra of $\overline{\mathcal{G}}$. Hence we have the following embeddings:

$$h_3 \subset h_4 \subset \mathcal{S} \qquad sl(2, \mathbb{R}) \subset gl(2) \subset \mathcal{S} \qquad \mathcal{G} \subset \overline{\mathcal{G}} \subset \mathcal{S}. \tag{1.2}$$

It is well-known that $\mathcal{S}$ arises as the Lie-point symmetry algebra of the $(1+1)$ heat-Schrödinger equation (SE). This relationship has motivated the search for q-deformed analogues of $\mathcal{S}$ related to space-time discretizations of the SE on geometric lattices of the type $x_n = q^n x_0$. In [3, 4] q-deformations of the vector field realization of $\mathcal{S}$ were considered as symmetry algebras of different discretized versions of the SE. From another point of view, a different q-Schrödinger algebra was directly introduced in [5] in order to obtain a generalized q-SE from its deformed representation theory. However, none of these q-algebras has been found to be consistent with a Hopf algebra structure. On the other hand, a discretization of the SE on a regular space-time lattice $x_n = x_0 + nz$ has been also studied from a symmetry approach in [6]. In this case, the resultant symmetry operators close a non-deformed Schrödinger algebra. Other q-SEs, which are related with quantum algebras different to $\mathcal{S}$, can be found in [7, 8, 9].

The first Hopf algebra deformations of $\mathcal{S}$ were introduced in [10, 11]. These two quantum algebras were obtained by starting from (non-standard) quantum deformation of certain subalgebras, namely, h_4 [12] and $gl(2)$ [13, 14, 15]. Furthermore these quantum $\mathcal{S}$ algebras lead to discretizations of the SE on uniform lattices [16]. Recently it was shown in [17] that, through a suitable non-linear change of basis, the algebra sector of the two Schrödinger Hopf algebras [10, 11] can be mapped onto its classical counterpart, thus preserving the deformation entirely in the coalgebra sector. In this way the discretized SEs of [16] were directly related with the ones previously obtained in [6].

All these results suggest the systematic investigation of all the possible quantum Schrödinger algebras that can be endowed with a Hopf algebra structure. This problem leads, as a necessary first order approximation, to the obtention and classification of all the Schrödinger Lie bialgebras. This is the main goal of the paper. We obtain in Section 2 the Lie bialgebra structures associated with the extended Schrödinger algebra in $(1+1)$ dimensions. As $\mathcal{S}$ is a non-semisimple Lie algebra our procedure will consist in computing the most general cocommutator and, independently, the most general classical r-matrix for $\mathcal{S}$; thus the coboundary Schrödinger bialgebras can be identified. The final result is that, despite the non-semisimple nature of $\mathcal{S}$, *all* its Lie bialgebra structures are coboundary ones, that is, they can be always obtained from a classical r-matrix. This enables us to construct their corresponding Poisson–Lie groups by making use of the Sklyanin bracket.

The following sections are devoted to show that this result contains a great amount of useful information concerning quantum deformations of the Schrödinger algebra. As a first step, in Section 3 we impose one generator to be primitive (besides M) in order to obtain explicitly some relevant families of Lie bialgebras from the previous classification. In particular, we are lead to three different types of Lie bialgebras, that will be further divided into two subfamilies each (standard and non-standard). Section 4 is devoted to the investigation of the embeddings of h_4, $gl(2)$ and $\overline{\mathcal{G}}$ Lie bialgebra structures into the Schrödinger classification. This study provides a systematic method to analyse the existence of the extension of some known quantum deformations of all these relevant subalgebras to the full Schrödinger algebra.

As an application of this embedding technique, Section 5 presents two new examples of q-deformed Schrödinger algebras endowed with a Hopf algebra structure. In both cases, the universal quantum R-matrices can be also obtained starting from the subalgebras. Finally, some remarks concerning further applications of quantum Schrödinger algebras are pointed out.

2 The Schrödinger Lie bialgebras

A *Lie bialgebra* (g, δ) is a Lie algebra g endowed with a linear map $\delta : g \to g \otimes g$ (the *cocommutator*) such that:

i) δ is a 1-cocycle, i.e.,

$$\delta([X,Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \qquad \forall X, Y \in g. \tag{2.1}$$

ii) The dual map $\delta^* : g^* \otimes g^* \to g^*$ is a Lie bracket on g^*.

A Lie bialgebra (g, δ) is called a *coboundary* Lie bialgebra if there exists an (skew-symmetric) element $r \in g \otimes g$ (the *classical r-matrix*) such that

$$\delta(X) = [1 \otimes X + X \otimes 1,\, r] \qquad \forall X \in g. \tag{2.2}$$

Coboundary Lie bialgebras can be of two different types:

i) *Non-standard* (or triangular): The r-matrix is a skew-symmetric solution of the classical Yang–Baxter equation (YBE):

$$[[r, r]] = 0 \tag{2.3}$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{2.4}$$

If we denote $r = r^{ij} X_i \otimes X_j$, then $r_{12} = r^{ij} X_i \otimes X_j \otimes 1, r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$.

ii) *Standard* (or quasitriangular): The r-matrix is a skew-symmetric solution of the modified classical YBE:

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X,\, [[r, r]]\,] = 0 \qquad \forall X \in g. \tag{2.5}$$

Finally, two Lie bialgebras (g, δ) and (g, δ') are said to be *equivalent* if there exists an automorphism O of g such that $\delta' = (O \otimes O) \circ \delta \circ O^{-1}$.

2.1 The general solution

The procedure to characterize all Schrödinger Lie bialgebras has two main steps. First it is necessary to deduce the most general cocommutator, and second, to find out which of the bialgebras so obtained come from a classical r-matrix. In order not to burden the exposition, computational details are given in the appendix, and the final result is summed up in the following.

Theorem 2.1. *All (1+1)-dimensional centrally extended Schrödinger Lie bialgebras are coboundary ones. They are defined by a classical r-matrix, $r \in \mathcal{S} \wedge \mathcal{S}$, which depends on 15 (real) coefficients a_i, b_i, c_j ($i = 1, \ldots, 6$; $j = 1, 2, 3$):*

$$\begin{aligned} r = {} & a_1 D \wedge P + a_2 D \wedge H + a_3 P \wedge M + a_4 H \wedge M + a_5 P \wedge H + a_6 P \wedge C \\ & + b_1 D \wedge K + b_2 D \wedge C + b_3 K \wedge M + b_4 C \wedge M + b_5 K \wedge C + b_6 K \wedge H \\ & + c_1 D \wedge M + c_2 P \wedge K + c_3 H \wedge C \end{aligned} \tag{2.6}$$

where the bialgebra coefficients are subjected to 19 equations casted into three sets:

$$\begin{aligned} & a_6^2 + a_6 b_1 - 3a_1 b_5 + b_5 b_6 = 0 \\ & a_2 a_3 - 2a_1 a_4 - a_4 b_6 - 3a_5 c_1 + a_5 c_2 = 0 \\ & a_1 a_2 + a_2 b_6 - a_5 c_3 = 0 \\ & a_5 b_1 - a_1 b_6 - 2a_2 c_1 - a_4 c_3 = 0 \\ & a_4 a_6 + a_4 b_1 - a_2 b_3 - a_5 b_4 + a_1 c_1 - a_1 c_2 = 0 \\ & 3a_1 b_2 - a_2 b_5 + a_6 c_3 + b_1 c_3 = 0 \\ & a_3 b_2 + 2a_1 b_4 - a_4 b_5 + a_6 c_1 + a_6 c_2 + b_3 c_3 = 0 \\ & 3a_2 b_5 + b_2 b_6 - a_6 c_3 = 0 \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&b_6^2 + b_6 a_1 - 3b_1 a_5 + a_5 a_6 = 0\\
&b_2 b_3 - 2b_1 b_4 - b_4 a_6 - 3b_5 c_1 - b_5 c_2 = 0\\
&b_1 b_2 + b_2 a_6 + b_5 c_3 = 0\\
&b_5 a_1 - b_1 a_6 - 2b_2 c_1 + b_4 c_3 = 0\\
&b_4 b_6 + b_4 a_1 - b_2 a_3 - b_5 a_4 + b_1 c_1 + b_1 c_2 = 0\\
&3b_1 a_2 - b_2 a_5 - b_6 c_3 - a_1 c_3 = 0\\
&b_3 a_2 + 2b_1 a_4 - b_4 a_5 + b_6 c_1 - b_6 c_2 - a_3 c_3 = 0\\
&3b_2 a_5 + a_2 a_6 + b_6 c_3 = 0
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&4a_2 b_2 + c_3^2 = 0\\
&2a_4 b_2 + 2a_2 b_4 + a_5 b_5 - a_6 b_6 = 0\\
&2a_1 b_1 - a_1 a_6 - b_1 b_6 + a_5 b_5 - a_6 b_6 = 0.
\end{aligned}
\tag{2.9}
$$

The Schouten bracket of r (2.6) turns out to be

$$
[[r, r]] = (a_3 a_6 + b_3 b_6 - a_3 b_1 - a_1 b_3 - c_2^2) K \wedge M \wedge P \tag{2.10}
$$

and the two types of classical r-matrices are distinguished by an additional equation:

$$
\begin{array}{ll}
\textit{Standard:} & a_3 a_6 + b_3 b_6 - a_3 b_1 - a_1 b_3 - c_2^2 \neq 0\\
\textit{Non-standard:} & a_3 a_6 + b_3 b_6 - a_3 b_1 - a_1 b_3 - c_2^2 = 0.
\end{array}
\tag{2.11}
$$

The classical r-matrix (2.6) gives rise to the general cocommutators by applying the relation (2.2):

$$
\begin{aligned}
\delta(D) &= -a_1 D \wedge P - 2a_2 D \wedge H - a_3 P \wedge M - 2a_4 H \wedge M - 3a_5 P \wedge H + a_6 P \wedge C\\
&\quad + b_1 D \wedge K + 2b_2 D \wedge C + b_3 K \wedge M + 2b_4 C \wedge M + 3b_5 K \wedge C - b_6 K \wedge H\\
\delta(P) &= (a_6 - b_1) K \wedge P - a_2 H \wedge P - b_1 D \wedge M - b_2 (C \wedge P + D \wedge K)\\
&\quad - b_4 K \wedge M + b_5 C \wedge M + b_6 H \wedge M + (c_1 - c_2) P \wedge M + c_3 K \wedge H\\
\delta(K) &= (a_1 - b_6) P \wedge K + b_2 C \wedge K + a_1 D \wedge M + a_2 (H \wedge K + D \wedge P)\\
&\quad + a_4 P \wedge M - a_5 H \wedge M - a_6 C \wedge M - (c_1 + c_2) K \wedge M + c_3 P \wedge C\\
\delta(H) &= -(2a_1 + b_6) P \wedge H - (a_6 + b_1) D \wedge P - 2b_1 K \wedge H - 2b_2 C \wedge H\\
&\quad - b_3 P \wedge M + b_4 D \wedge M - b_5 (D \wedge K + P \wedge C) + 2c_1 H \wedge M - c_3 D \wedge H\\
\delta(C) &= (2b_1 + a_6) K \wedge C + (b_6 + a_1) D \wedge K + 2a_1 P \wedge C + 2a_2 H \wedge C\\
&\quad + a_3 K \wedge M - a_4 D \wedge M + a_5 (D \wedge P + K \wedge H) - 2c_1 C \wedge M - c_3 D \wedge C\\
\delta(M) &= 0.
\end{aligned}
\tag{2.12}
$$

The most general element $\eta \in \mathcal{S} \otimes \mathcal{S}$ which is $Ad^{\otimes 2}$ invariant is simply $\eta = \tau M \otimes M$ where τ is an arbitrary real number. The classical r-matrix $r' = r + \eta$ gives rise to the same Schrödinger Lie bialgebra than r, that is, (2.12); hence r' is the most general non-skewsymmetric classical r-matrix for $\mathcal{S}$.

On the other hand, the Schrödinger algebra automorphism defined by

$$
\begin{array}{lll}
D \to -D & P \to -K & K \to -P\\
M \to -M & H \to -C & C \to -H
\end{array}
\tag{2.13}
$$

(which leaves the Lie brackets (1.1) invariant) can be implemented at a bialgebra level by introducing a suitable transformation of the parameters a_i, b_i and c_j given by

$$\begin{array}{lll} a_i \to b_i & b_i \to a_i & i = 1, \dots, 6 \\ c_1 \to c_1 & c_2 \to -c_2 & c_3 \to -c_3 . \end{array} \tag{2.14}$$

We stress that under the bialgebra automorphism defined by the maps (2.13) and (2.14), the general classical r-matrix (2.6), the equations (2.9) and the Schouten bracket (2.10) remain invariant, while the sets of equations (2.7) and (2.8) are interchanged. As expected, we also find that $\delta(P) \leftrightarrow \delta(K)$, $\delta(H) \leftrightarrow \delta(C)$, while $\delta(D)$ and $\delta(M)$ remain unchanged.

2.2 The Schrödinger Poisson–Lie groups

Since all Schrödinger Lie bialgebras are coboundary ones, we can obtain their corresponding Poisson–Lie groups by means of the Sklyanin bracket provided by a given classical r-matrix $r = \sum_{i,j} r^{ij} X_i \otimes X_j$ [18]:

$$\{f, g\} = \sum_{i,j} r^{ij} (X_i^L f \, X_j^L g - X_i^R f \, X_j^R g) \tag{2.15}$$

where X_i^L and X_j^R are left and right invariant vector fields on the Schrödinger group. Thus we consider the following 4×4 real matrix representation of the centrally extended Schrödinger algebra [19]:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.16}$$

This allows us to write an element of the extended Schrödinger group as

$$\begin{aligned} g &= \exp\{mM\} \exp\{pP\} \exp\{kK\} \exp\{hH\} \exp\{cC\} \exp\{dD\} \\ &= \begin{pmatrix} 1 & (k - (p + kh)c)e^{-d} & (p + kh)e^{d} & 2m - pk \\ 0 & (1 - hc)e^{-d} & he^{d} & p \\ 0 & -ce^{-d} & e^{d} & -k \\ 0 & 0 & 0 & 1 \end{pmatrix} . \end{aligned} \tag{2.17}$$

Then left and right invariant vector fields can be deduced

$$X_H^L = e^{-2d} \left\{ \partial_h - c\partial_d - c^2 \partial_c \right\}$$

$$
\begin{aligned}
X_P^L &= (1-ch)e^{-d}\{\partial_p + k\partial_m\} + ce^{-d}\partial_k \\
X_K^L &= e^d\{\partial_k - h\partial_p - hk\partial_m\} \\
X_D^L &= \partial_d \qquad X_C^L = e^{2d}\partial_c \qquad X_M^L = \partial_m
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
X_H^R &= \partial_h - k\partial_p - \frac{1}{2}k^2\partial_m \qquad X_P^R = \partial_p \\
X_K^R &= \partial_k + p\partial_m \qquad X_M^R = \partial_m \\
X_D^R &= \partial_d + 2c\partial_c - 2h\partial_h - p\partial_p + k\partial_k \\
X_C^R &= -h\partial_d + (1-2hc)\partial_c + h^2\partial_h + p\partial_k + \frac{1}{2}p^2\partial_m.
\end{aligned}
\tag{2.19}
$$

We substitute now these vector fields and the r-matrix (2.6) within the Sklyanin bracket (2.15) and compute the Poisson–Lie brackets amongst the group coordinates $\{d, h, p, k, c, m\}$; they turn out to be

$$
\begin{aligned}
\{d,h\} &= a_2(e^{-2d}-1) + b_2h^2 - c_3h \\
\{d,p\} &= a_1\left(e^{-d}(1-ch)-1\right) + a_2k + a_5ce^{-3d}(1-ch) - a_6h \\
&\quad -b_1he^d + b_2hp - b_6hce^{-d} + c_3hk \\
\{d,k\} &= a_1ce^{-d} + a_5c^2e^{-3d} + b_1(e^d-1) - b_2(p+hk) - b_5h + b_6ce^{-d} \\
\{d,c\} &= -a_2c^2e^{-2d} + b_2(e^{2d}-1) - c_3c \\
\{h,p\} &= 2a_1h - a_2(p+2hk) + a_5\left(1-e^{-3d}(1-ch)\right) + a_6h^2 \\
&\quad -b_2ph^2 + b_6he^{-d} - c_3kh^2 \\
\{h,k\} &= a_2k - a_5ce^{-3d} + 2b_1h + b_2h(2p+hk) + b_5h^2 + b_6(1-e^{-d}) - c_3p \\
\{h,c\} &= 2a_2c + 2b_2h(1-ch) + 2c_3hc \\
\{p,k\} &= a_1k - a_2k^2 - a_6p + b_1p + b_2p^2 - b_6k + c_3kp \\
\{p,c\} &= 2a_1c - 2a_2kc - a_5c^2e^{-3d}(1-ch) + a_6\left(e^d(1-ch) - 1 + 2ch\right) \\
&\quad +b_2p(1-2ch) - b_5he^{3d} + b_6hc^2e^{-d} + c_3k(1-2ch) \\
\{k,c\} &= -a_5c^3e^{-3d} + a_6ce^d + 2b_1c - b_2\left(k(1-2ch) - 2cp\right) \\
&\quad +b_5(e^{3d} - 1 + 2ch) - b_6c^2e^{-d} \qquad\qquad (2.20) \\
\{m,d\} &= -a_1ke^{-d}(1-ch) - \frac{1}{2}a_2k^2 + a_4ce^{-2d} - a_5kce^{-3d}(1-ch) \\
&\quad +b_1(p+hke^d) + \frac{1}{2}b_2p^2 - b_4h + b_5hp + b_6hkce^{-d} - \frac{1}{2}c_3hk^2 \\
\{m,h\} &= a_2hk^2 + a_4(1-e^{-2d}) + a_5ke^{-3d}(1-ch) - 2b_1hp - b_2hp^2 \\
&\quad +b_4h^2 - b_5ph^2 - b_6(p+hke^{-d}) - 2c_1h + \frac{1}{2}c_3(p^2+h^2k^2) \\
\{m,p\} &= \frac{1}{2}a_2pk^2 + a_3\left(1-e^{-d}(1-ch)\right) - a_4k - \frac{1}{2}a_5k^2 + \frac{1}{2}a_6p^2 \\
&\quad -b_1p^2 - \frac{1}{2}b_2p^3 + b_3he^d + b_6kp - c_1p + c_2p - \frac{1}{2}c_3kp^2 \\
\{m,k\} &= -\frac{1}{2}a_2k^3 - a_3ce^{-d} + b_1kp + \frac{1}{2}b_2kp^2 - b_3(e^d-1) + b_4p
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}b_5p^2 - \frac{1}{2}b_6k^2 + c_1k + c_2k + \frac{1}{2}c_3pk^2 \\
\{m,c\} = & -a_2ck^2 + a_4c^2e^{-2d} - a_5kc^2e^{-3d}(1-ch) + a_6ke^d(1-ch) \\
& +2b_1cp + b_2cp^2 - b_4(e^{2d} - 1 + 2ch) - b_5\left(hke^{3d} + p(1-2ch)\right) \\
& +b_6hkc^2e^{-d} + 2c_1c + \frac{1}{2}c_3k^2(1-2ch).
\end{aligned}
$$

The parameters a_i, b_i and c_j must satisfy the 19 relations (2.7)–(2.9); it is a matter of cumbersome computations to recover these conditions from the Jacobi identities coming from the Poisson algebra (2.20). On the other hand, it can be checked that, as expected, the linear terms of these Poisson–Lie brackets lead to the dual of the cocommutators (2.12).

3 The Schrödinger Lie bialgebras with two primitive generators

A common feature of all extended Schrödinger bialgebras is that the central generator M is always *primitive*, i.e., its cocommutator vanishes $\delta(M) = 0$; this, in turn, means that its coproduct can be always taken as $\Delta(M) = 1 \otimes M + M \otimes 1$ after deformation. In general, the existence of primitive generators strongly determines as much mathematical as possible physical properties of the corresponding quantum deformation (see for example [10, 11]). Therefore, in order to specialize our general results into several multiparametric families and to obtain explicit quantization results, we impose *one* additional generator X to be primitive (besides M). Furthermore, the condition $\delta(X) = 0$ rather simplifies the equations (2.7)–(2.11). Finally, due to the automorphisms (2.13) and (2.14), there is no loss of generality if we restrict our study to three types of bialgebras: those with either D, P or H primitive.

3.1 D primitive

The condition $\delta(D) = 0$ leaves c_1, c_2 and c_3 as the initial free parameters, all the remaining ones being equal to zero. The equations (2.7)–(2.9) imply that $c_3 = 0$, so that the Schouten bracket reduces to $[[r,r]] = -c_2^2 K \wedge M \wedge P$. Hence, the condition $c_2 \neq 0$ leads to the standard subfamily of Lie bialgebras, while $c_2 = 0$ gives the non-standard one. Therefore each subfamily of Lie bialgebras is characterized by the following parameters:

- Standard subfamily: c_1, $c_2 \neq 0$.
- Non-standard subfamily: c_1.

From now on, it will be understood that all the parameters that do not appear explicitly for each subfamily are equal to zero. We will also write the constraints that the non-vanishing parameters must satisfy.

The resulting r-matrix, cocommutators and non-vanishing Poisson–Lie brackets for these subfamilies are given by

$$\begin{aligned}
&r = c_1 D\wedge M + c_2 P\wedge K\\
&\delta(D) = 0 \qquad \delta(M) = 0\\
&\delta(P) = (c_1 - c_2) P\wedge M \qquad \delta(K) = -(c_1 + c_2) K\wedge M\\
&\delta(H) = 2c_1 H\wedge M \qquad \delta(C) = -2c_1 C\wedge M.
\end{aligned} \tag{3.1}$$

$$\{m,h\} = -2c_1 h \qquad \{m,p\} = -(c_1 - c_2)p \qquad \{m,k\} = (c_1+c_2)k \qquad \{m,c\} = 2c_1 c. \tag{3.2}$$

Obviously, the non-standard Lie bialgebra corresponds to the substitution $c_2 = 0$ in these expressions.

3.2 P primitive

If we impose now $\delta(P) = 0$, we find that the initial parameters are: a_1, a_3, a_4, a_5, b_3 and c_1 with $c_2 = c_1$; all the others vanish. The equations (2.7)–(2.9) reduce to a single relation: $a_1 a_4 + a_5 c_1 = 0$. The Schouten bracket $[[r,r]] = -(a_1 b_3 + c_1^2) K\wedge M\wedge P$ characterizes the standard and non-standard subfamilies:

• Standard subfamily: a_1, a_3, a_4, a_5, b_3, c_1 with $c_2 = c_1$, $a_1 a_4 + a_5 c_1 = 0$ and $a_1 b_3 + c_1^2 \neq 0$.

• Non-standard subfamily: a_1, a_3, a_4, a_5, b_3, c_1 with $c_2 = c_1$, $a_1 a_4 + a_5 c_1 = 0$ and $a_1 b_3 + c_1^2 = 0$.

The final Lie bialgebra structure and Poisson–Lie brackets for both subfamilies are given by:

$$\begin{aligned}
&r = a_1 D\wedge P + a_3 P\wedge M + a_4 H\wedge M + a_5 P\wedge H + b_3 K\wedge M\\
&\qquad\quad + c_1(D\wedge M + P\wedge K)\\
&\delta(P) = 0 \qquad \delta(M) = 0\\
&\delta(D) = -a_1 D\wedge P - a_3 P\wedge M - 2a_4 H\wedge M - 3a_5 P\wedge H + b_3 K\wedge M\\
&\delta(K) = a_1(P\wedge K + D\wedge M) + a_4 P\wedge M - a_5 H\wedge M - 2c_1 K\wedge M\\
&\delta(H) = -2a_1 P\wedge H - b_3 P\wedge M + 2c_1 H\wedge M\\
&\delta(C) = a_1(D\wedge K + 2P\wedge C) + a_3 K\wedge M - a_4 D\wedge M + a_5(D\wedge P + K\wedge H)\\
&\qquad\quad -2c_1 C\wedge M.
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\{d,h\} = 0 \qquad \{d,p\} = a_1\left(e^{-d}(1-ch) - 1\right) + a_5 c e^{-3d}(1-ch)\\
&\{d,k\} = a_1 c e^{-d} + a_5 c^2 e^{-3d} \qquad \{d,c\} = 0\\
&\{h,p\} = 2a_1 h + a_5\left(1 - e^{-3d}(1-ch)\right) \qquad \{h,k\} = -a_5 c e^{-3d} \qquad \{h,c\} = 0\\
&\{p,k\} = a_1 k \qquad \{p,c\} = 2a_1 c - a_5 c^2 e^{-3d}(1-ch) \qquad \{k,c\} = -a_5 c^3 e^{-3d}\\
&\{m,d\} = -a_1 k e^{-d}(1-ch) + a_4 c e^{-2d} - a_5 k c e^{-3d}(1-ch)
\end{aligned}$$

$$
\begin{aligned}
\{m,h\} &= a_4(1-e^{-2d}) + a_5 k e^{-3d}(1-ch) - 2c_1 h \\
\{m,p\} &= a_3\left(1-e^{-d}(1-ch)\right) - a_4 k - \frac{1}{2}a_5 k^2 + b_3 h e^d \\
\{m,k\} &= -a_3 c e^{-d} - b_3(e^d-1) + 2c_1 k \\
\{m,c\} &= a_4 c^2 e^{-2d} - a_5 k c^2 e^{-3d}(1-ch) + 2c_1 c.
\end{aligned} \tag{3.4}
$$

We recall that the quantum Schrödinger algebra whose underlying non-standard Lie bialgebra corresponds to setting $a_1 = -z$, $a_3 = z/2$ and the remaining parameters equal to zero was obtained in [10]. Furthermore it was proven in [16, 17] that a space discretization of the $(1+1)$ heat-Schrödinger equation on a uniform lattice is endowed with this Hopf algebra symmetry.

3.3 H primitive

Finally, $\delta(H) = 0$ renders a_1, a_2, a_3, a_4, a_5 and c_2 with $b_6 = -2a_1$ as free parameters. Moreover the set of equations (2.7)–(2.9) implies that $a_1 = 0$ and $a_2 a_3 + a_5 c_2 = 0$. The Schouten bracket in this case is given by $[[r,r]] = -c_2^2 K \wedge M \wedge P$ so that the standard solution reads

- Standard subfamily: a_2, a_3, a_4, $c_2 \neq 0$ with $a_5 = -a_2 a_3 / c_2$.

$$
\begin{aligned}
& r = a_2 D \wedge H + a_3 P \wedge M + a_4 H \wedge M - \frac{a_2 a_3}{c_2} P \wedge H + c_2 P \wedge K \\
& \delta(H) = 0 \qquad \delta(M) = 0 \qquad \delta(P) = -a_2 H \wedge P - c_2 P \wedge M \\
& \delta(D) = -2a_2 D \wedge H - a_3 P \wedge M - 2a_4 H \wedge M + 3\frac{a_2 a_3}{c_2} P \wedge H \\
& \delta(K) = a_2(H \wedge K + D \wedge P) + a_4 P \wedge M + \frac{a_2 a_3}{c_2} H \wedge M - c_2 K \wedge M \\
& \delta(C) = 2a_2 H \wedge C + a_3 K \wedge M - a_4 D \wedge M - \frac{a_2 a_3}{c_2}(D \wedge P + K \wedge H).
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
& \{d,h\} = a_2(e^{-2d}-1) \qquad \{d,p\} = a_2 k - \frac{a_2 a_3}{c_2} c e^{-3d}(1-ch) \\
& \{d,k\} = -\frac{a_2 a_3}{c_2} c^2 e^{-3d} \qquad \{d,c\} = -a_2 c^2 e^{-2d} \\
& \{h,p\} = -a_2(p + 2hk) - \frac{a_2 a_3}{c_2}\left(1 - e^{-3d}(1-ch)\right) \\
& \{h,k\} = a_2 k + \frac{a_2 a_3}{c_2} c e^{-3d} \qquad \{h,c\} = 2a_2 c \qquad \{p,k\} = -a_2 k^2 \\
& \{p,c\} = -2a_2 k c + \frac{a_2 a_3}{c_2} c^2 e^{-3d}(1-ch) \qquad \{k,c\} = \frac{a_2 a_3}{c_2} c^3 e^{-3d} \\
& \{m,d\} = -\frac{1}{2}a_2 k^2 + a_4 c e^{-2d} + \frac{a_2 a_3}{c_2} k c e^{-3d}(1-ch) \\
& \{m,h\} = a_2 h k^2 + a_4(1-e^{-2d}) - \frac{a_2 a_3}{c_2} k e^{-3d}(1-ch)
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\{m,p\} &= \frac{1}{2}a_2pk^2 + a_3\left(1-e^{-d}(1-ch)\right) - a_4k + \frac{a_2a_3}{2c_2}k^2 + c_2p \\
\{m,k\} &= -\frac{1}{2}a_2k^3 - a_3ce^{-d} + c_2k \\
\{m,c\} &= -a_2ck^2 + a_4c^2e^{-2d} + \frac{a_2a_3}{c_2}kc^2e^{-3d}(1-ch).
\end{aligned}
$$

One arrives at the non-standard subfamily by requiring $c_2 = 0$, that is, with parameters $\{a_2, a_3, a_4, a_5\}$, together with the constraint $a_2a_3 = 0$. If we assume $a_2 = 0$, then $\delta(P) = 0$ and we are within the non-standard subfamily of Section 3.2. Therefore, we have not to take into account this possibility, and consider the case $a_3 = 0$ as follows:

- Non-standard subfamily: a_2, a_4, a_5.

$$
\begin{aligned}
& r = a_2D\wedge H + a_4H\wedge M + a_5P\wedge H \\
& \delta(H) = 0 \qquad \delta(M) = 0 \qquad \delta(P) = -a_2H\wedge P \\
& \delta(D) = -2a_2D\wedge H - 2a_4H\wedge M - 3a_5P\wedge H \\
& \delta(K) = a_2(H\wedge K + D\wedge P) + a_4P\wedge M - a_5H\wedge M \\
& \delta(C) = 2a_2H\wedge C - a_4D\wedge M + a_5(D\wedge P + K\wedge H).
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
& \{d,h\} = a_2(e^{-2d}-1) \qquad \{d,p\} = a_2k + a_5ce^{-3d}(1-ch) \\
& \{d,k\} = a_5c^2e^{-3d} \qquad \{d,c\} = -a_2c^2e^{-2d} \\
& \{h,p\} = -a_2(p+2hk) + a_5\left(1-e^{-3d}(1-ch)\right) \\
& \{h,k\} = a_2k - a_5ce^{-3d} \qquad \{h,c\} = 2a_2c \qquad \{p,k\} = -a_2k^2 \\
& \{p,c\} = -2a_2kc - a_5c^2e^{-3d}(1-ch) \qquad \{k,c\} = -a_5c^3e^{-3d} \\
& \{m,d\} = -\frac{1}{2}a_2k^2 + a_4ce^{-2d} - a_5kce^{-3d}(1-ch) \\
& \{m,h\} = a_2hk^2 + a_4(1-e^{-2d}) + a_5ke^{-3d}(1-ch) \\
& \{m,p\} = \frac{1}{2}a_2pk^2 - a_4k - \frac{1}{2}a_5k^2 \qquad \{m,k\} = -\frac{1}{2}a_2k^3 \\
& \{m,c\} = -a_2ck^2 + a_4c^2e^{-2d} - a_5kc^2e^{-3d}(1-ch).
\end{aligned}
\tag{3.8}
$$

The quantum deformation of the non-standard Lie bialgebra corresponding to $a_2 = -2z$, $a_4 = z$ and $a_5 = 0$ was constructed in [11]; this leads to a time discretization of the heat-Schrödinger equation on a uniform lattice with Hopf algebra symmetry [11, 17].

4 Lie sub-bialgebra embeddings

In this section we investigate the embeddings (1.2) at a Lie bialgebra level, that is, we analyse which of the harmonic oscillator h_4 [12], extended Galilei $\overline{\mathcal{G}}$ [20] and

$gl(2)$ [21] Lie bialgebras can be embedded within the Schrödinger bialgebras as *Lie sub-bialgebras*. In order to state precisely this idea let us consider two Lie bialgebras (h, δ_h) and (g, δ_g) in such a manner that h is a Lie subalgebra of g: $h \subset g$. We say that h is a Lie sub-bialgebra of g if the cocommutator δ_g in g of any generator $X_i \in h$ is of the form

$$\delta_g(X_i) = f_i^{jk} X_j \wedge X_k \qquad X_j, X_k \in h. \tag{4.1}$$

As a consequence, the following results establish whether it is possible or not to construct a quantum Schrödinger algebra with one of the above subalgebras as a Hopf subalgebra. In other words, whenever the Lie bialgebra embedding exists for a given subalgebra, one can make use of its known quantum deformations in order to obtain a quantum Schrödinger algebra (this was the procedure used for two particular deformations in [10, 11]).

4.1 h_4 Lie sub-bialgebras

The commutation rules of the harmonic oscillator algebra h_4 in the usual basis $\{N, A_+, A_-, M\}$ are given by

$$[N, A_+] = A_+ \qquad [N, A_-] = -A_- \qquad [A_-, A_+] = M \qquad [M, \cdot\,] = 0. \tag{4.2}$$

It was proven in [22] that all the h_4 Lie bialgebras are coboundary ones; all of them, together with their quantum deformations, were explicitly obtained in [12]. The general classical r-matrix for h_4, which depends on six parameters $\{\alpha_+, \alpha_-, \beta_+, \beta_-, \vartheta, \xi\}$, is given by

$$r = \alpha_+ N \wedge A_+ + \alpha_- N \wedge A_- + \vartheta N \wedge M + \xi A_+ \wedge A_- + \beta_+ A_+ \wedge M + \beta_- A_- \wedge M \tag{4.3}$$

where the parameters must fulfil

$$\alpha_+ \alpha_- = 0 \qquad \alpha_+(\xi + \vartheta) = 0 \qquad \alpha_-(\xi - \vartheta) = 0 \tag{4.4}$$

and the Schouten bracket reads

$$[[r, r]] = (\alpha_+ \beta_- + \alpha_- \beta_+ - \xi^2) M \wedge A_+ \wedge A_-. \tag{4.5}$$

The cocommutators, which can be obtained from (2.2), turn out to be

$$\begin{aligned}
&\delta(N) = \alpha_+ N \wedge A_+ - \alpha_- N \wedge A_- + \beta_+ A_+ \wedge M - \beta_- A_- \wedge M \\
&\delta(A_+) = -\alpha_-(N \wedge M + A_+ \wedge A_-) - (\vartheta + \xi) A_+ \wedge M \\
&\delta(A_-) = \alpha_+(N \wedge M - A_+ \wedge A_-) + (\vartheta - \xi) A_- \wedge M \\
&\delta(M) = 0.
\end{aligned} \tag{4.6}$$

The h_4 algebra can be considered as a subalgebra of $\mathcal{S}$ (1.1) by denoting the oscillator generators as

$$N \to -D \qquad A_+ \to P \qquad A_- \to K \qquad M \to M. \tag{4.7}$$

Now we write (4.6) in terms of $\{D, P, K, M\}$ and according to (4.1) impose that their general cocommutators (2.12) give exactly (4.6). This requirement implies that the Schrödinger bialgebra parameters must be

$$\begin{array}{lll} a_1 = -\alpha_+ & a_3 = \beta_+ & c_1 = -\vartheta \\ b_1 = -\alpha_- & b_3 = \beta_- & c_2 = \xi \end{array} \tag{4.8}$$

with all the remaining ones equal to zero. Under these assumptions it can be checked that the full set of 19 equations (2.7)–(2.9) reduces to (4.4), the Schrödinger r-matrix (2.6) is equal to (4.3) and the Schouten bracket (2.10) reproduces (4.5). Hence we conclude that

Proposition 4.1. *All harmonic oscillator Lie bialgebras with generators $\{D, P, K, M\}$ are Schrödinger Lie sub-bialgebras. The resulting Schrödinger classical r-matrix and cocommutators containing the h_4 sub-bialgebras depend on six parameters $\{\alpha_+, \alpha_-, \beta_+, \beta_-, \vartheta, \xi\}$ subjected to (4.4); they are*

$$\begin{array}{l} r = -\alpha_+ D \wedge P - \alpha_- D \wedge K + \beta_+ P \wedge M + \beta_- K \wedge M - \vartheta D \wedge M + \xi P \wedge K \\[1ex] \delta(D) = \alpha_+ D \wedge P - \alpha_- D \wedge K - \beta_+ P \wedge M + \beta_- K \wedge M \\ \delta(P) = \alpha_-(D \wedge M - P \wedge K) - (\vartheta + \xi) P \wedge M \\ \delta(K) = -\alpha_+(D \wedge M + P \wedge K) + (\vartheta - \xi) K \wedge M \\ \delta(M) = 0 \\ \delta(H) = 2\alpha_+ P \wedge H + \alpha_-(D \wedge P + 2K \wedge H) - \beta_- P \wedge M - 2\vartheta H \wedge M \\ \delta(C) = -2\alpha_- K \wedge C - \alpha_+(D \wedge K + 2P \wedge C) + \beta_+ K \wedge M + 2\vartheta C \wedge M. \end{array} \tag{4.9}$$

The Schouten bracket is $[[r, r]] = (\alpha_+\beta_- + \alpha_-\beta_+ - \xi^2) K \wedge M \wedge P$.

4.2 $gl(2)$ Lie sub-bialgebras

Now, we consider the $gl(2)$ Lie algebra whose generators $\{J_3, J_+, J_-, I\}$ satisfy the following commutation rules:

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = J_3 \qquad [I, \cdot\,] = 0 \tag{4.10}$$

where I is the central generator. All the $gl(2)$ Lie bialgebras are coboundary ones coming from a classical r-matrix which depends on six parameters $\{a_+, a_-, b_+, b_-, a, b\}$ [21]:

$$r = \frac{1}{2}(a_+ J_3 \wedge J_+ - a_- J_3 \wedge J_- - bJ_3 \wedge I + b_+ J_+ \wedge I - b_- J_- \wedge I - 2aJ_+ \wedge J_-) \tag{4.11}$$

which is subjected to the relations

$$a_+ b - b_+ a = 0 \qquad a_+ b_- + a_- b_+ = 0 \qquad a_- b + b_- a = 0. \tag{4.12}$$

The Schouten bracket is given by

$$[[r, r]] = (a^2 + a_+ a_-) J_3 \wedge J_+ \wedge J_-. \tag{4.13}$$

The corresponding cocommutators are

$$\begin{aligned}
&\delta(J_3) = a_+ J_3 \wedge J_+ + a_- J_3 \wedge J_- + b_+ J_+ \wedge I + b_- J_- \wedge I\\
&\delta(J_+) = a J_3 \wedge J_+ - \frac{1}{2} b_- J_3 \wedge I + a_- J_+ \wedge J_- + b J_+ \wedge I\\
&\delta(J_-) = a J_3 \wedge J_- - \frac{1}{2} b_+ J_3 \wedge I - a_+ J_+ \wedge J_- - b J_- \wedge I\\
&\delta(I) = 0.
\end{aligned} \tag{4.14}$$

The $gl(2)$ algebra can be regarded as a subalgebra of $\mathcal{S}$ once we rename the $gl(2)$ generators as:

$$J_3 \to -D \qquad J_+ \to H \qquad J_- \to -C \qquad I \to M. \tag{4.15}$$

We demand now that the Schrödinger cocommutators (2.12) for $\{D, H, C, M\}$ lead to (4.14); thus the Schrödinger bialgebra parameters turn out to be

$$\begin{array}{lll}
a_2 = -\frac{1}{2} a_+ & a_4 = \frac{1}{2} b_+ & c_1 = \frac{1}{2} b\\
b_2 = -\frac{1}{2} a_- & b_4 = \frac{1}{2} b_- & c_3 = a
\end{array} \tag{4.16}$$

with c_2 arbitrary and all the others equal to zero. In this case, the equations (2.7)–(2.9) give rise to (4.12), together with

$$a^2 + a_+ a_- = 0. \tag{4.17}$$

This last equation implies that the Schouten bracket of $gl(2)$ vanishes, which in turn precludes the embedding of the standard $gl(2)$ bialgebras. These results can be summarized as follows:

Proposition 4.2. *Amongst all the $gl(2)$ Lie bialgebras with generators $\{D, H, C, M\}$, only the non-standard ones are Schrödinger Lie sub-bialgebras. The Schrödinger r-matrix and cocommutators which comprise the non-standard $gl(2)$ sub-bialgebras depend on seven parameters $\{a_+, a_-, b_+, b_-, a, b, c_2\}$ which fulfil (4.12) and (4.17):*

$$\begin{aligned}
r = \tfrac{1}{2}(&-a_+ D \wedge H - a_- D \wedge C + b_+ H \wedge M + b_- C \wedge M\\
&+ b D \wedge M + 2a H \wedge C + 2c_2 P \wedge K)
\end{aligned}$$

$$\begin{aligned}
&\delta(D) = a_+ D \wedge H - a_- D \wedge C - b_+ H \wedge M + b_- C \wedge M\\
&\delta(H) = -a D \wedge H + \tfrac{1}{2} b_- D \wedge M - a_- H \wedge C + b H \wedge M\\
&\delta(C) = -a D \wedge C - \tfrac{1}{2} b_+ D \wedge M - a_+ H \wedge C - b C \wedge M\\
&\delta(M) = 0\\
&\delta(P) = \tfrac{1}{2}(a_+ H \wedge P + a_-(C \wedge P + D \wedge K) - b_- K \wedge M\\
&\qquad\qquad + (b - 2c_2) P \wedge M + 2a K \wedge H)\\
&\delta(K) = -\tfrac{1}{2}(a_- C \wedge K + a_+(H \wedge K + D \wedge P) - b_+ P \wedge M\\
&\qquad\qquad + (b + 2c_2) K \wedge M - 2a P \wedge C).
\end{aligned} \tag{4.18}$$

The Schouten bracket is $[[r, r]] = -c_2^2 K \wedge M \wedge P$.

This statement shows that there exist both standard Schrödinger Lie bialgebras ($c_2 \neq 0$) and non-standard ones ($c_2 = 0$) that contain non-standard $gl(2)$ Lie sub-bialgebras. On the other hand, let us recall that in [5], a q-deformed Schrödinger

algebra was obtained by demanding that the subalgebra structure of (standard) $sl_q(2)$ should be preserved by the deformation. This led to a deformation of $\mathcal{S}$ for which the previous analysis proves that no Hopf structure can be constructed. In other words, there cannot exist any Hopf algebra deformation of $\mathcal{S}$ retaining the standard $sl_q(2)$ as a Hopf subalgebra. This example enlightens the usefulness of a Lie bialgebra approach when looking for a precise quantum deformation of a given Lie algebra.

4.3 $\overline{\mathcal{G}}$ Lie sub-bialgebras

In the usual kinematical basis, the $(1+1)$-dimensional extended Galilei algebra $\overline{\mathcal{G}}$ is spanned by $\{K, H, P, M\}$, directly arising as a subalgebra of $\mathcal{S}$ (1.1). The cocommutators of the $\overline{\mathcal{G}}$ Lie bialgebras can be written collectively in terms of nine parameters $\{\alpha, \xi, \nu, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ as [20] (see also [23]):

$$\begin{aligned}
&\delta(K) = \beta_6 K \wedge P + \xi K \wedge M + \nu P \wedge H + \beta_1 P \wedge M + \beta_2 H \wedge M\\
&\delta(H) = \beta_5 K \wedge M - (\beta_6 + \alpha) P \wedge H + \beta_3 P \wedge M + (\beta_4 - \xi) H \wedge M\\
&\delta(P) = \beta_4 P \wedge M + (\beta_6 + \alpha) H \wedge M\\
&\delta(M) = \alpha P \wedge M
\end{aligned} \tag{4.19}$$

satisfying the following equations:

$$\begin{aligned}
&\alpha\beta_5 = 0 \qquad \beta_6(\beta_6 + \alpha) = 0 \qquad \beta_4(\beta_6 + \alpha) = 0\\
&\nu(\xi - \beta_4) = 0 \qquad \alpha(\xi - \beta_4) - \nu\beta_5 = 0.
\end{aligned} \tag{4.20}$$

Unlike the h_4 and $gl(2)$ cases, the $\overline{\mathcal{G}}$ Lie bialgebras include both non-coboundary and coboundary Lie bialgebras [20]. The standard $\overline{\mathcal{G}}$ bialgebras depend on two parameters $\{\xi \neq 0, \beta_1\}$ with $\beta_4 = \xi$ while the others are equal to zero; the corresponding classical r-matrix reads

$$r = \xi K \wedge P + \beta_1 H \wedge M. \tag{4.21}$$

The non-standard ones correspond to take the three parameters $\{\beta_1, \beta_2, \beta_3\}$ and the r-matrix is given by

$$r = \beta_1 H \wedge M + \beta_2 H \wedge P + \beta_3 M \wedge K. \tag{4.22}$$

We impose the Schrödinger cocommutators (2.12) to be equal to (4.19); this implies that α, ν, β_5 must vanish, while the Schrödinger parameters are

$$\begin{aligned}
&a_4 = \beta_1 \qquad a_5 = -\beta_2 \qquad c_1 = \tfrac{1}{2}(\beta_4 - \xi)\\
&b_3 = -\beta_3 \qquad b_6 = \beta_6 \qquad c_2 = -\tfrac{1}{2}(\beta_4 + \xi)
\end{aligned} \tag{4.23}$$

with a_3 arbitrary and the remaining ones equal to zero. The equations (2.7)–(2.9) lead to $\beta_6 = 0$ and

$$\beta_2(2\beta_4 - \xi) = 0. \tag{4.24}$$

Therefore the equations (4.20) are trivially satisfied. The final result is summed up by

Proposition 4.3. *Amongst all the extended Galilei Lie bialgebras with generators $\{K, H, P, M\}$ and cocommutators given by (4.19), only those with $\alpha = \nu = \beta_5 = \beta_6 = 0$ are Schrödinger Lie sub-bialgebras. The Schrödinger Lie bialgebras which contain the $\overline{\mathcal{G}}$ sub-bialgebras depend on six parameters $\{\xi, \beta_1, \beta_2, \beta_3, \beta_4, a_3\}$ fulfilling (4.24):*

$$
\begin{aligned}
r &= a_3 P\wedge M + \beta_1 H\wedge M - \beta_2 P\wedge H - \beta_3 K\wedge M \\
&\qquad\qquad + \tfrac{1}{2}(\beta_4-\xi) D\wedge M - \tfrac{1}{2}(\beta_4+\xi) P\wedge K \\
\delta(K) &= \xi K\wedge M + \beta_1 P\wedge M + \beta_2 H\wedge M \\
\delta(H) &= \beta_3 P\wedge M + (\beta_4-\xi) H\wedge M \\
\delta(P) &= \beta_4 P\wedge M \\
\delta(M) &= 0 \\
\delta(D) &= -a_3 P\wedge M - 2\beta_1 H\wedge M + 3\beta_2 P\wedge H - \beta_3 K\wedge M \\
\delta(C) &= a_3 K\wedge M - \beta_1 D\wedge M - \beta_2 (D\wedge P + K\wedge H) - (\beta_4-\xi) C\wedge M.
\end{aligned}
\tag{4.25}
$$

The Schouten bracket is $[[r, r]] = -\frac{1}{4}(\beta_4+\xi)^2 K\wedge M\wedge P$.

Notice that *all* the coboundary $\overline{\mathcal{G}}$ bialgebras determined by either (4.21) or (4.22) are Schrödinger sub-bialgebras. There are also non-coboundary $\overline{\mathcal{G}}$ bialgebras arising as Schrödinger sub-bialgebras.

5 Quantum Schrödinger algebras and quantum universal R-matrices

The main advantage of a systematic analysis of Lie bialgebra structures on a given Lie algebra is that they characterize possible 'directions' for the obtention of quantum deformations which are guided by the cocommutator δ, that gives the first order in the deformation of the comultiplication map. As we have already mentioned, there are only two quantum $\mathcal{S}$ algebras endowed with a known Hopf structure [10, 11], and both of them are of non-standard type. In this section we focus mainly on the standard $\mathcal{S}$ bialgebras with a two-fold purpose. First, we construct new quantum Schrödinger algebras starting from Lie bialgebras with either D or H as primitive generators. Second, we illustrate with these examples the Lie sub-bialgebra embeddings studied in the previous section and exploit them in order to deduce universal R-matrices for these quantum $\mathcal{S}$ algebras.

5.1 D primitive: $U_{c_1,c_2}(\mathcal{S})$

Let us consider the Schrödinger bialgebras with D primitive that are defined through (3.1). The corresponding two-parameter Hopf algebra $U_{c_1,c_2}(\mathcal{S})$ has the following coproduct and commutation rules:

$$
\begin{aligned}
&\Delta(D) = 1\otimes D + D\otimes 1 \qquad \Delta(M) = 1\otimes M + M\otimes 1 \\
&\Delta(P) = 1\otimes P + P\otimes e^{(c_1-c_2)M} \qquad \Delta(K) = 1\otimes K + K\otimes e^{-(c_1+c_2)M}
\end{aligned}
$$

$$\Delta(H) = 1 \otimes H + H \otimes e^{2c_1 M} \qquad \Delta(C) = 1 \otimes C + C \otimes e^{-2c_1 M} \tag{5.1}$$

$$\begin{array}{lll} [D,P] = -P & [D,K] = K & [K,P] = \dfrac{1-e^{-2c_2 M}}{2c_2} \\ [D,H] = -2H & [D,C] = 2C & [H,C] = D \\ [K,H] = P & [K,C] = 0 & [M,\cdot\,] = 0 \\ [P,C] = -K & [P,H] = 0. & \end{array} \tag{5.2}$$

The counit is trivial and the antipode can be easily deduced from the Hopf algebra axioms. Notice that $U_{c_1,c_2}(\mathcal{S})$ is a standard quantum algebra whenever $c_2 \neq 0$, while it leads to a non-standard one under the limit $c_2 \to 0$, $U_{c_1}(\mathcal{S})$, which has non-deformed commutation rules.

By taking into account the results presented in section 4, it can be checked that the Lie bialgebra (3.1) has h_4, $gl(2)$ and $\overline{\mathcal{G}}$ Lie sub-bialgebras. After deformation, these structures give rise to Hopf subalgebras of $U_{c_1,c_2}(\mathcal{S})$, namely

(i) A standard quantum algebra $U_{\vartheta,\xi}(h_4)$ of type II of [12], with generators $\{D, P, K, M\}$ and $\vartheta = -c_1$, $\xi = c_2$.

(ii) A non-standard quantum algebra $U_b(gl(2))$ belonging to the family II of [21], with generators $\{D, H, C, M\}$ and $b = 2c_1$.

(iii) A non-coboundary quantum algebra $U_{\xi,\beta_4}(\overline{\mathcal{G}})$ of the family Ia of [20], with generators $\{K, H, P, M\}$ and $\xi = -c_1 - c_2$, $\beta_4 = c_1 - c_2$.

In what follows we show how the above quantum subalgebras allow us to obtain directly universal R-matrices for either $U_{c_2}(\mathcal{S})$ or $U_{c_1}(\mathcal{S})$.

We consider the standard algebra $U_{c_2}(\mathcal{S})$ by fixing $c_1 = 0$ in (5.1) and (5.2). It contains a standard quantum extended Galilei subalgebra $U_\xi(\overline{\mathcal{G}})$ (note that in this case $\beta_4 = \xi = -c_2$) whose quantum universal R-matrix was deduced in [20]. In the Schrödinger basis this element reads

$$\begin{aligned} \mathcal{R} &= \exp\{c_2 P \wedge K f(M, c_2)\} \\ f(M, c_2) &= \frac{e^{c_2 M/2} \otimes e^{c_2 M/2}}{\sqrt{\sinh c_2 M \otimes \sinh c_2 M}} \arcsin\left(\frac{\sqrt{\sinh c_2 M \otimes \sinh c_2 M}}{\cosh((c_2/2)\Delta(M))}\right). \end{aligned} \tag{5.3}$$

As it was proven in [20], $\mathcal{R}$ is not a solution of the quantum YBE but it fulfils

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X), \tag{5.4}$$

where $\sigma(X \otimes Y) = Y \otimes X$, for the Galilei generators $\{K, H, P, M\}$. Furthermore, since

$$[c_2 P \wedge K f(M, c_2), 1 \otimes X + X \otimes 1] = 0 \quad \text{for} \quad X = \{D, C\} \tag{5.5}$$

the relation (5.4) is also satisfied by D and C so that (5.3) is a non-quasitriangular universal R-matrix for $U_{c_2}(\mathcal{S})$. Notice that its underlying classical r-matrix is $r = c_2 P \wedge K$.

On the other hand, the non-standard quantum algebra $U_{c_1}(\mathcal{S})$ provided by $c_2 \to 0$ comprises a non-standard quantum subalgebra $U_b(gl(2))$ whose universal R-matrix

was obtained in [21]:

$$\mathcal{R} = \exp\{-c_1 M \otimes D\} \exp\{c_1 D \otimes M\} \tag{5.6}$$

which not only verifies the relation (5.4) but also the quantum YBE for $\{D, H, C, M\}$. The two remaining generators P and K also fulfil (5.4):

$$\begin{aligned} &\exp\{c_1 D \otimes M\}\Delta(X) \exp\{-c_1 D \otimes M\} = 1 \otimes X + X \otimes 1 \equiv \Delta_0(X) \\ &\exp\{-c_1 M \otimes D\}\Delta_0(X) \exp\{c_1 M \otimes D\} = \sigma \circ \Delta(X) \quad \text{for } X = \{P, K\}. \end{aligned} \tag{5.7}$$

Consequently, the element (5.6) is a triangular universal R-matrix for $U_{c_1}(\mathcal{S})$ with classical r-matrix given by $r = c_1 D \wedge M$.

5.2 H primitive: $U_{a_2,c_2}(\mathcal{S})$

As a second example, we consider the standard $\mathcal{S}$ bialgebra with H primitive (3.5) with two parameters a_2, $c_2 \neq 0$ while a_3, a_4 are taken to be zero:

$$\begin{aligned} &r = a_2 D \wedge H + c_2 P \wedge K \\ &\delta(H) = 0 \qquad \delta(M) = 0 \\ &\delta(D) = -2a_2 D \wedge H \qquad \delta(P) = P \wedge (a_2 H - c_2 M) \\ &\delta(C) = -2a_2 C \wedge H \qquad \delta(K) = K \wedge (-a_2 H - c_2 M) + a_2 D \wedge P. \end{aligned} \tag{5.8}$$

The quantum deformation of this bialgebra leads to a two-parameter Hopf algebra $U_{a_2,c_2}(\mathcal{S})$ endowed with the following coproduct and commutation rules:

$$\begin{aligned} &\Delta(H) = 1 \otimes H + H \otimes 1 \qquad \Delta(M) = 1 \otimes M + M \otimes 1 \\ &\Delta(D) = 1 \otimes D + D \otimes e^{-2a_2 H} \qquad \Delta(C) = 1 \otimes C + C \otimes e^{-2a_2 H} \\ &\Delta(P) = 1 \otimes P + P \otimes e^{a_2 H} e^{-c_2 M} \\ &\Delta(K) = 1 \otimes K + K \otimes e^{-a_2 H} e^{-c_2 M} + a_2 D \otimes e^{-2a_2 H} P \end{aligned} \tag{5.9}$$

$$\begin{aligned} &[D, P] = -P \qquad [D, K] = K \qquad [K, P] = \frac{1 - e^{-2c_2 M}}{2c_2} \\ &[D, H] = \frac{e^{-2a_2 H} - 1}{a_2} \qquad [D, C] = 2C - a_2 D^2 \qquad [H, C] = D \\ &[K, H] = e^{-2a_2 H} P \qquad [K, C] = -\frac{1}{2} a_2 (KD + DK) \qquad [M, \cdot\,] = 0 \\ &[P, C] = -K + \frac{1}{2} a_2 (DP + PD) \qquad [P, H] = 0. \end{aligned} \tag{5.10}$$

This quantum algebra is of standard type whenever $c_2 \neq 0$; however if we perfom the limit $c_2 \to 0$ in (5.9) and (5.10), then we obtain a non-standard quantum algebra $U_{a_2}(\mathcal{S})$ whose Lie bialgebra belongs to (3.7).

The generators $\{D, H, C, M\}$ close a non-standard Hopf subalgebra

$$U_{a_+}(gl(2)) = U_{a_+}(sl(2, \mathbb{R})) \oplus u(1) \subset U_{a_2,c_2}(\mathcal{S})$$

belonging to the family I_+ of [21], where $a_+ = -2a_2$, $u(1)$ corresponds to the central generator M, and $U_{a_+}(sl(2,\mathbb{R}))$ is the well-known Jordanian or non-standard quantum $sl(2,\mathbb{R})$ algebra [24, 25, 26, 27, 28] writen in the basis of [29]. The quantum universal R-matrix of $U_{a_+}(sl(2,\mathbb{R}))$ [29] (see also [30]) can be expressed in the Schrödinger basis as

$$\mathcal{R} = \exp\{-a_2 H \otimes D\} \exp\{a_2 D \otimes H\}. \tag{5.11}$$

In this respect, we stress that the quantum universal R-matrix for $U_{a_+}(sl(2,\mathbb{R}))$ was firstly obtained by Ogievetsky through a twist operator [31]. The element $\mathcal{R}$ is a solution of the quantum YBE and also fulfils (5.4) for $\{D, H, C, M\}$. If we restrict now to the non-standard quantum algebra $U_{a_2}(\mathcal{S})$ by taking $c_2 \to 0$, it can be checked that the generators P and K also satisfy the relation (5.4) by use of similar equations to (5.7). Hence we conclude that (5.11) is a triangular universal R-matrix for $U_{a_2}(\mathcal{S})$ whose corresponding classical r-matrix is $r = a_2 D \wedge H$.

6 Concluding remarks

This paper presents the explicit computation and analysis of all the possible Lie bialgebra structures associated to the Schrödinger algebra, a non-semisimple Lie algebra of dimension 6. In general, this kind of classification is shown to be feasible and enables a systematic study of all possible quantum deformations of a given Lie algebra. In particular, this kind of construction simplifies quite efficiently the task of finding a certain deformation that should fit with some fixed properties that are known *a priori*.

In particular, the coboundary nature of all the Schrödinger Lie bialgebras is a remarkable result that certainly enhances the structural vicinity of this Lie algebra with respect to semisimple ones. On the other hand, the richness of the subalgebra structure of $\mathcal{S}$ allows to show a great variety of situations concerning the generalization of Lie sub-bialgebras to full Lie bialgebras on $\mathcal{S}$. In this respect, we find all kinds of situations: sometimes, standard Lie sub-bialgebras can be generalized, and sometimes they cannot; moreover, even non-coboundary Lie sub-bialgebras can be generalized to coboundary ones in the bigger Lie algebra. These examples suggest that the embedding of a given non-semisimple algebra within a higher dimensional one is an interesting procedure that could provide some new information connecting quantum deformations of both algebras.

On the other hand, we would like to recall that the two-photon algebra h_6 [32], generated by the operators $\{N, A_+, A_-, B_+, B_-, M\}$ and endowed with the following commutation rules

$$\begin{array}{lll} [N, A_+] = A_+ & [N, A_-] = -A_- & [A_-, A_+] = M \\ [N, B_+] = 2B_+ & [N, B_-] = -2B_- & [B_-, B_+] = 4N + 2M \\ [A_+, B_-] = -2A_- & [A_+, B_+] = 0 & [M, \cdot\,] = 0 \\ [A_-, B_+] = 2A_+ & [A_-, B_-] = 0 & \end{array} \tag{6.1}$$

is isomorphic to $\mathcal{S}$ through the map

$$D = -N - \frac{1}{2}M \qquad P = A_+ \qquad K = A_- \qquad H = \frac{1}{2}B_+ \qquad C = \frac{1}{2}B_-. \tag{6.2}$$

Therefore, all the results contained in this paper can be immediately translated into the two-photon algebra language through (6.2), and the new quantum deformations that have been constructed could be interpreted in a completely different physical framework. Recall that the two-photon algebra is the dynamical symmetry algebra of the single-mode radiation field Hamiltonian that describes in a unified setting coherent, squeezed and intelligent states of light [33]. Thus, the results here presented provide a preliminary basis for the application of quantum deformations in the construction of non-classical states of light (see, for instance, [10, 11] where deformed Fock–Bargmann realizations corresponding to two different quantum deformations of the two-photon/Schrödinger algebra were introduced).

Finally, we would like to emphasize that quantum algebras and completely integrable systems are directly related through the formalism introduced in [34]. In particular, the Hamiltonian

$$\mathcal{H}_{gl(2)} = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \mathcal{G}\left(\sum_{i=1}^{N} m_i q_i^2\right), \tag{6.3}$$

where $\mathcal{G}$ is an arbitrary function, can be constructed through the Poisson realization of the $gl(2) = \{N, B_+, B_-, M\}$ coalgebra, and completely integrable (and superintegrable) deformations of (6.3) were given through some quantum deformations of $gl(2)$ in [35]. On the other hand, completely integrable deformations of Hamiltonians of the type

$$\mathcal{H}_{\overline{\mathcal{G}}} = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \mathcal{F}\left(\sum_{i=1}^{N} m_i q_i\right), \tag{6.4}$$

where $\mathcal{F}$ is again arbitrary, can be obtained through quantum Poisson $\overline{\mathcal{G}}$ coalgebras (with generators $\overline{\mathcal{G}} = \{B_+, A_+, A_-, M\}$) [20].

Since both $gl(2)$ and $\overline{\mathcal{G}}$ are subalgebras of $\mathcal{S}$, these constructions can be generalized by making use of Schrödinger coalgebras. In fact, by following [34], it can be shown that the non-deformed Schrödinger coalgebra gives rise to the N-particle Hamiltonian

$$\mathcal{H}_{\mathcal{S}} = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \mathcal{F}\left(\sum_{i=1}^{N} m_i q_i\right) + \mathcal{G}\left(\sum_{i=1}^{N} m_i q_i^2\right) \tag{6.5}$$

through the following phase space realization of $\mathcal{S}$

$$N = qp - \frac{m}{2} \qquad A_+ = p \qquad A_- = mq \qquad M = m \qquad B_+ = \frac{p^2}{m} \qquad B_- = mq^2. \tag{6.6}$$

Integrals of the motion for (6.5) can be obtained by using the coproducts of the (fourth-order) Casimir of $\mathcal{S}$ [19]. From this perspective, each quantum deformation

of the Schrödinger algebra provides an integrable deformation of the Hamiltonian (6.5). In order to obtain such deformations explicitly, the corresponding deformed Casimir and phase space realization is needed. A comprehensive treatment of the integrability properties linked to (classical and quantum) Schrödinger coalgebras will be given elsewhere.

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Appendix: proof of Theorem 2.1

As $\mathcal{S}$ is a six-dimensional Lie algebra, the most general cocommutator $\delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ will be a linear combination (with 6×15 real coefficients) of skewsymmetric products of the generators X_i of $\mathcal{S}$:

$$\delta(X_i) = f_i^{jk}\, X_j \wedge X_k \qquad j < k \qquad i,j,k = 1,\ldots,6 \tag{A.1}$$

which in matrix form reads

$$\delta \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} = \begin{pmatrix} f_1^{12} & \ldots & f_1^{16} & f_1^{23} & \ldots & f_1^{56} \\ f_2^{12} & \ldots & f_2^{16} & f_2^{23} & \ldots & f_2^{56} \\ f_3^{12} & \ldots & f_3^{16} & f_3^{23} & \ldots & f_3^{56} \\ f_4^{12} & \ldots & f_4^{16} & f_4^{23} & \ldots & f_4^{56} \\ f_5^{12} & \ldots & f_5^{16} & f_5^{23} & \ldots & f_5^{56} \\ f_6^{12} & \ldots & f_6^{16} & f_6^{23} & \ldots & f_6^{56} \end{pmatrix} \begin{pmatrix} X_1 \wedge X_2 \\ \vdots \\ X_1 \wedge X_6 \\ X_2 \wedge X_3 \\ \vdots \\ X_5 \wedge X_6 \end{pmatrix}. \tag{A.2}$$

We choose

$$X_1 = D \qquad X_2 = C \qquad X_3 = H \qquad X_4 = K \qquad X_5 = P \qquad X_6 = M \tag{A.3}$$

and impose the cocycle condition (2.1) onto the arbitrary expression (A.2). This implies that the initial 90 coefficients f_i^{jk} can be expressed in terms of 15 independent parameters that we denote α_l ($l = 1,\ldots,15$), namely

$$\begin{array}{lllll} f_1^{12} = \alpha_1 & f_1^{13} = \alpha_2 & f_1^{14} = \alpha_3 & f_1^{15} = \alpha_4 & f_2^{12} = \alpha_5 \\ f_2^{26} = \alpha_6 & f_1^{24} = \alpha_7 & f_1^{25} = \alpha_8 & f_1^{26} = \alpha_9 & f_1^{34} = \alpha_{10} \\ f_1^{35} = \alpha_{11} & f_1^{36} = \alpha_{12} & f_4^{46} = \alpha_{13} & f_1^{46} = \alpha_{14} & f_1^{56} = \alpha_{15}. \end{array} \tag{A.4}$$

The remaining non-vanishing coefficients read

$$\begin{array}{llll} f_2^{14} = \alpha_{10} - \alpha_4 & f_2^{15} = \alpha_{11}/3 & f_2^{16} = \alpha_{12}/2 & f_2^{23} = \alpha_2 \\ f_2^{24} = \alpha_8 - 2\alpha_3 & f_2^{25} = 2\alpha_4 & f_2^{34} = -\alpha_{11}/3 & f_2^{46} = -\alpha_{15} \end{array}$$

$$
\begin{array}{lllll}
f_3^{13} = \alpha_5 & f_3^{14} = \alpha_7/3 & f_3^{15} = \alpha_8 - \alpha_3 & f_3^{16} = \alpha_9/2 & \\
f_3^{23} = -\alpha_1 & f_3^{25} = -\alpha_7/3 & f_3^{34} = 2\alpha_3 & f_3^{35} = \alpha_{10} - 2\alpha_4 & \\
f_3^{36} = -\alpha_6 & f_3^{56} = -\alpha_{14} & f_4^{15} = -\alpha_2/2 & f_4^{16} = -\alpha_4 & f_4^{24} = \alpha_1/2 \\
f_4^{25} = \alpha_5 & f_4^{26} = \alpha_8 & f_4^{34} = -\alpha_2/2 & f_4^{36} = -\alpha_{11}/3 & \\
f_4^{45} = \alpha_{10} + \alpha_4 & f_4^{56} = -\alpha_{12}/2 & f_5^{14} = -\alpha_1/2 & f_5^{16} = -\alpha_3 & \\
f_5^{25} = -\alpha_1/2 & f_5^{26} = -\alpha_7/3 & f_5^{34} = \alpha_5 & f_5^{35} = \alpha_2/2 & \\
f_5^{36} = \alpha_{10} & f_5^{45} = -(\alpha_3 + \alpha_8) & f_5^{46} = -\alpha_9/2 & f_5^{56} = \alpha_{13} - \alpha_6. &
\end{array}
\tag{A.5}
$$

Hence we have obtained the following cocommutator:

$$
\begin{aligned}
\delta(D) &= \alpha_1 D \wedge C + \alpha_2 D \wedge H + \alpha_3 D \wedge K + \alpha_4 D \wedge P \\
&\quad + \alpha_7 C \wedge K + \alpha_8 C \wedge P + \alpha_9 C \wedge M + \alpha_{10} H \wedge K \\
&\quad + \alpha_{11} H \wedge P + \alpha_{12} H \wedge M + \alpha_{14} K \wedge M + \alpha_{15} P \wedge M \\
\delta(C) &= \alpha_5 D \wedge C + (\alpha_{10} - \alpha_4) D \wedge K + \frac{\alpha_{11}}{3} D \wedge P \\
&\quad + \frac{\alpha_{12}}{2} D \wedge M + \alpha_2 C \wedge H + (\alpha_8 - 2\alpha_3) C \wedge K + 2\alpha_4 C \wedge P \\
&\quad + \alpha_6 C \wedge M - \frac{\alpha_{11}}{3} H \wedge K - \alpha_{15} K \wedge M \\
\delta(H) &= \alpha_5 D \wedge H + \frac{\alpha_7}{3} D \wedge K + (\alpha_8 - \alpha_3) D \wedge P \\
&\quad + \frac{\alpha_9}{2} D \wedge M - \alpha_1 C \wedge H - \frac{\alpha_7}{3} C \wedge P + 2\alpha_3 H \wedge K \\
&\quad + (\alpha_{10} - 2\alpha_4) H \wedge P - \alpha_6 H \wedge M - \alpha_{14} P \wedge M \\
\delta(K) &= -\frac{\alpha_2}{2} D \wedge P - \alpha_4 D \wedge M + \frac{\alpha_1}{2} C \wedge K + \alpha_5 C \wedge P \\
&\quad + \alpha_8 C \wedge M - \frac{\alpha_2}{2} H \wedge K - \frac{\alpha_{11}}{3} H \wedge M \\
&\quad + (\alpha_{10} + \alpha_4) K \wedge P + \alpha_{13} K \wedge M - \frac{\alpha_{12}}{2} P \wedge M \\
\delta(P) &= -\frac{\alpha_1}{2} D \wedge K - \alpha_3 D \wedge M - \frac{\alpha_1}{2} C \wedge P - \frac{\alpha_7}{3} C \wedge M \\
&\quad + \alpha_5 H \wedge K + \frac{\alpha_2}{2} H \wedge P + \alpha_{10} H \wedge M - \frac{\alpha_9}{2} K \wedge M \\
&\quad - (\alpha_8 + \alpha_3) K \wedge P + (\alpha_{13} - \alpha_6) P \wedge M \\
\delta(M) &= 0.
\end{aligned}
\tag{A.6}
$$

Next, we impose Jacobi identities onto the dual map $\delta^* : \mathcal{S}^* \otimes \mathcal{S}^* \to \mathcal{S}^*$ in order to ensure that δ^* is a Lie bracket on $\mathcal{S}^*$. This condition gives rise to 19 equations

$$
\begin{aligned}
&\alpha_8^2 - \alpha_3\alpha_8 - \alpha_4\alpha_7 - \alpha_7\alpha_{10}/3 = 0 \\
&3\alpha_2\alpha_{15} - 6\alpha_4\alpha_{12} + 4\alpha_6\alpha_{11} + 3\alpha_{10}\alpha_{12} - 2\alpha_{11}\alpha_{13} = 0 \\
&3\alpha_2\alpha_{10} - 3\alpha_2\alpha_4 - 2\alpha_5\alpha_{11} = 0 \\
&3\alpha_2\alpha_6 - 2\alpha_3\alpha_{11} - 6\alpha_4\alpha_{10} + 3\alpha_5\alpha_{12} = 0 \\
&\alpha_3\alpha_{12} - \alpha_2\alpha_{14} - 2\alpha_4\alpha_6 + 2\alpha_4\alpha_{13} - \alpha_8\alpha_{12} + \alpha_9\alpha_{11}/3 = 0 \\
&3\alpha_1\alpha_4 + 2\alpha_3\alpha_5 - 2\alpha_5\alpha_8 + \alpha_2\alpha_7/3 = 0 \\
&\alpha_1\alpha_{15} + 2\alpha_4\alpha_9 + 2\alpha_5\alpha_{14} - 2\alpha_8\alpha_{13} + \alpha_7\alpha_{12}/3 = 0 \\
&2\alpha_5\alpha_8 - \alpha_1\alpha_{10} - \alpha_2\alpha_7 = 0
\end{aligned}
\tag{A.7}
$$

$$
\alpha_{10}^2 - \alpha_3\alpha_{11} - \alpha_4\alpha_{10} - \alpha_8\alpha_{11}/3 = 0
$$

$$
\begin{aligned}
&3\alpha_1\alpha_{14} - 6\alpha_3\alpha_9 - 2\alpha_6\alpha_7 - 2\alpha_7\alpha_{13} + 3\alpha_8\alpha_9 = 0\\
&3\alpha_1\alpha_3 + 2\alpha_5\alpha_7 - 3\alpha_1\alpha_8 = 0\\
&3\alpha_1\alpha_6 + 6\alpha_3\alpha_8 + 2\alpha_4\alpha_7 - 3\alpha_5\alpha_9 = 0\\
&2\alpha_3\alpha_{13} - \alpha_1\alpha_{15} + \alpha_4\alpha_9 - \alpha_9\alpha_{10} + \alpha_7\alpha_{12}/3 = 0\\
&3\alpha_2\alpha_3 + 2\alpha_4\alpha_5 - 2\alpha_5\alpha_{10} + \alpha_1\alpha_{11}/3 = 0\\
&\alpha_2\alpha_{14} + 2\alpha_3\alpha_{12} + 2\alpha_5\alpha_{15} + 2\alpha_6\alpha_{10} - 2\alpha_{10}\alpha_{13} + \alpha_9\alpha_{11}/3 = 0\\
&2\alpha_5\alpha_{10} - \alpha_1\alpha_{11} - \alpha_2\alpha_8 = 0
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
&\alpha_1\alpha_2 - \alpha_5^2 = 0\\
&\alpha_1\alpha_{12} + \alpha_2\alpha_9 - 2\alpha_8\alpha_{10} + 2\alpha_7\alpha_{11}/9 = 0\\
&2\alpha_3\alpha_4 + \alpha_3\alpha_{10} + \alpha_4\alpha_8 - \alpha_8\alpha_{10} + \alpha_7\alpha_{11}/9 = 0.
\end{aligned}
\tag{A.9}
$$

Surprisingly, the relations (A.7), (A.8) and (A.9) can be mapped exactly and in this order onto the equations (2.7), (2.8) and (2.9) (which come from the modified classical YBE) by means of the following identification between the parameters α_l and a_i, b_i, c_j:

$$
\begin{array}{lllll}
\alpha_1 = 2b_2 & \alpha_2 = -2a_2 & \alpha_3 = b_1 & \alpha_4 = -a_1 & \alpha_5 = -c_3\\
\alpha_6 = -2c_1 & \alpha_7 = -3b_5 & \alpha_8 = -a_6 & \alpha_9 = 2b_4 & \alpha_{10} = b_6\\
\alpha_{11} = 3a_5 & \alpha_{12} = -2a_4 & \alpha_{13} = -(c_1 + c_2) & \alpha_{14} = b_3 & \alpha_{15} = -a_3.
\end{array}
\tag{A.10}
$$

Under this identification, the cocommutator (A.6) turns into the form (2.12). Consequently, all the Schrödinger Lie bialgebras are coboundary ones and can be completely described by a classical r-marix according to Theorem 2.1.

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